SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2007
Mathematics
Trial HSC

Mathematics (2 Unit)

General Instructions

- Reading Time – 5 Minutes
- Working time – 3 hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Start each new section of a separate writing booklet.
- The questions are **NOT** of equal value.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.
- Full marks will **NOT** be given unless the method of solution is shown.

Total Marks – 120

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

SECTION A

Start each section in a new answer booklet.

Question 1 [14 marks] **Marks**

(a) Find $\sqrt[3]{9.8^2}$ correct to 2 decimal places. 1

(b) Factorise $ax + 3ay - x - 3y$. 1

(c) Differentiate $\ln(5x)$. 1

(d) Express $\dfrac{2}{4+\sqrt{3}}$ with a rational denominator. 1

(e) Solve for a and d 1

$$a + 9d = 20$$
$$10(2a + 9d) = 120$$

(f) Find the exact value of $\dfrac{\sin 60°}{\cos 30°}$. 1

(g) Use the table of standard integrals to show that 2

$$\int_0^1 \frac{dx}{\sqrt{x^2+4}} = \ln\left(\frac{1+\sqrt{5}}{2}\right).$$

(h) Solve $\cos x = -\dfrac{1}{2}$ for $0 \le x \le 2\pi$. 2

(i) If $2 = x^{0.6}$ and $3 = x^{0.8}$, find $\log_x 6$. 2

(j) Solve $|3x - 1| = 5$. 2

Question 2 [12 marks] **Marks**

(a) Differentiate

(i) $\tan x$ 1

(ii) $(x^4+1)^{-2}$ 1

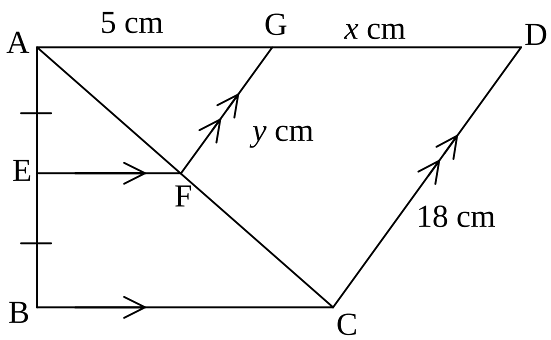
(b) For the circle $x^2+y^2-8x+6y+9=0$ find the coordinates of the centre and the radius. 2

(c)

(i) For $y=\dfrac{k}{x+2}$, where k is a constant, show that $\dfrac{dy}{dx}=\dfrac{-k}{(x+2)^2}$. 1

(ii) Find the value of k if the tangent to the curve $y=\dfrac{k}{x+2}$ at the point where $x=2$, has a gradient of $\dfrac{1}{4}$. 1

(d)



In the diagram above AE = EB, EF || BC, FG || CD, AG = 5cm and CD = 18cm. Find the value of x and y giving reasons. 2

(e) Find

(i) $\int \sin(2t - 1)\,dt$ 1

(ii) $\int_0^1 e^{2x}\,dx$ 1

(f) Solve $\log_{27} 16 = x \log_3 2$. 2

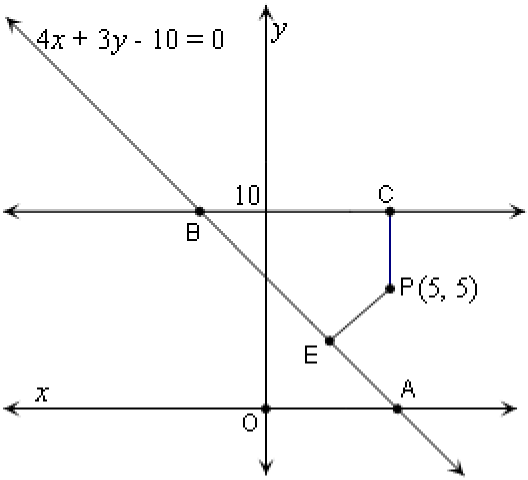
END OF SECTION A

SECTION B

Start a new answer booklet.

Question 3 [13 marks] **Marks**

(a)



In the diagram, the equations of the lines BE and BC are $4x + 3y - 10 = 0$ and $y = 10$ respectively. P is the point (5, 5), $PE \perp BE$ and $BC \perp PC$.

(i) Find the distance PC. 1

(ii) Show that the perpendicular distance from P to BE is 5 units. 1

(iii) Hence prove that $\Delta BCP \equiv \Delta BEP$. 3

(iv) Show that the coordinates of B are (–5, 10). 1

(v) What sort of quadrilateral is BCPE. 1

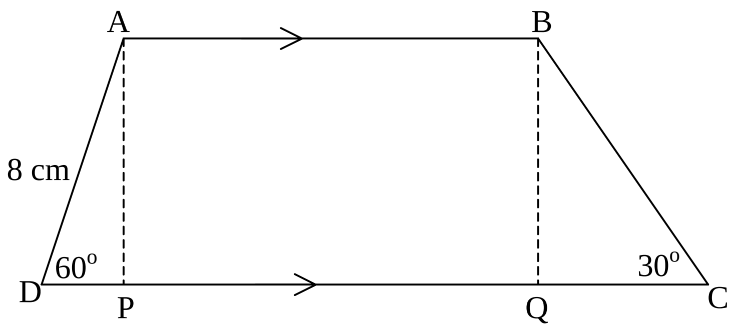
(vi) Find the area of BCPE. 1

(vii) Show that the locus of points which are equidistant from the lines BC and BE is given by the equation $x + 2y - 15 = 0$. 2

(b) Evaluate $\sum_{n=3}^{8} \left(2 \times 3^n - 2n\right)$. 3

Question 4 [12 marks] **Marks**

(a)



(i) Find the length of AP in exact form. 1

(ii) Hence, show that the length of QC is 12cm. 2

(b) From a pack of 52 playing cards, one card is drawn at random. Find the probability that it is,

(i) a king 1

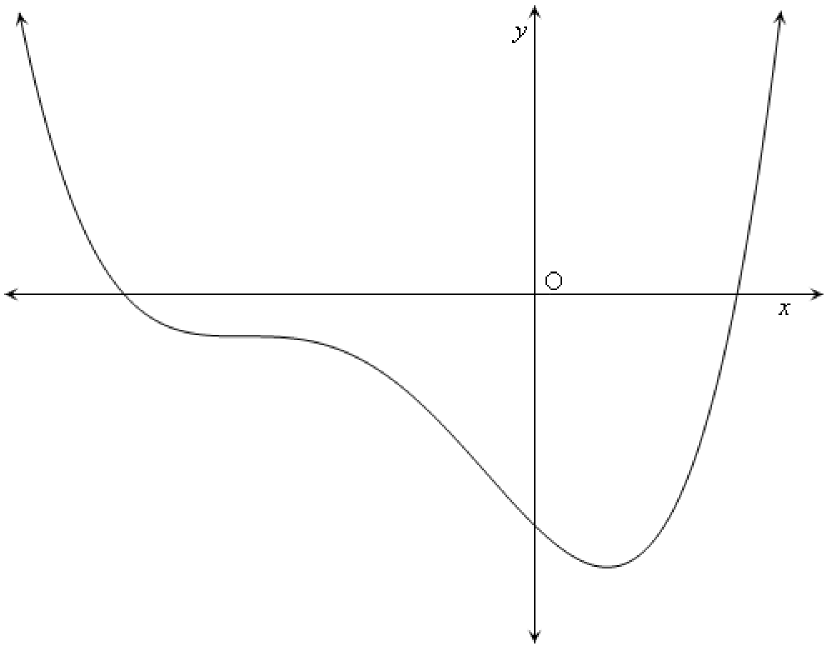
(ii) a red card 1

(iii) a red or a king 1

(c) Without sketching, determine if $y = 3x^2 - 4x + 5$ cuts the x-axis. 2

(d) Evaluate $\lim_{x \to 3} \frac{x^2 - 5x + 6}{x^2 - 9}$. 2

(e)



This is the graph of $y = f(x)$. Copy or trace this graph onto your answer booklet. On the same set of axes, sketch the graph of its derivative, $y = f'(x)$. 2

END OF SECTION B

SECTION C

Start a new answer booklet

Question 5 [14 marks] **Marks**

(a) For the series $486+324+216+144+...$

(i) Which term is $\frac{2048}{243}$? (Working must be shown.) 2

(ii) Does this series have a limiting sum? Give a reason for your answer. 1

(b) A polygon has 25 sides, the lengths of which form an arithmetic sequence.

(i) Find an expression for the perimeter in terms of the shortest side and the common difference. 1

(ii) The perimeter of the polygon is 1100cm and the longest side is 10 times the length of the shortest side. Find the length of the shortest side of the polygon and the common difference of the sequence. 3

(c) For the function $y=e^{-\frac{x^2}{2}}$,

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. 2

(ii) For what values of *x* is the curve of $y=e^{-\frac{x^2}{2}}$ concave down. 1

(d)

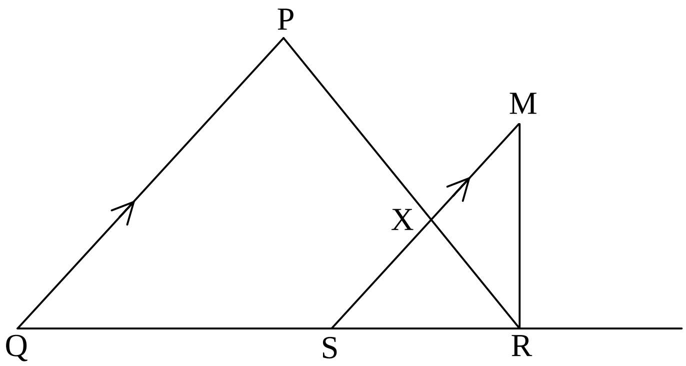
(i) Find the point of intersection of $y=2$ and $y=e^x$. 1

(ii) Indicate, by shading on a diagram, the region in the first quadrant bounded by the *y*-axis, the line $y=2$ and the curve $y=e^x$. 1

(iii) Use Simpson's Rule with 3 function values, to calculate the volume of the solid generated when the region in part (ii) is rotated about the *y*-axis (answer to 2 d.p.). 2

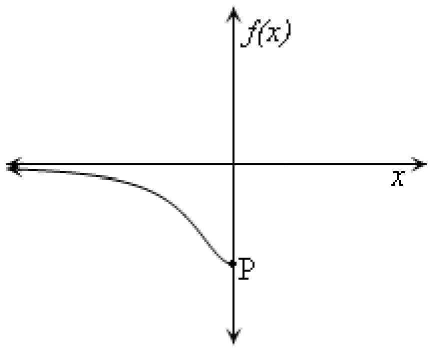
Question 6 [12 marks] **Marks**

(a)



In the diagram PQ = PR, XM = XR and PQ || MS. By letting $\angle MRX = \alpha°$. Show that $MR \perp QR$, giving full reasons. 3

(b) The diagram below shows part of the function $f(x) = \dfrac{-2}{1+x^2}$.



(i) Find the coordinates of point P on the diagram. 1

(ii) Prove that $f(x)$ is an even function. 1

(iii) Copy the diagram into your answer booklet and complete the curve. 1

(iv) State the range of this function. 1

(c) The volume, V litres, of a tank after t minutes is given by $V = 60 + 4t - t^2$.

(i) At what time(s) will the tank be empty? 1

(ii) Find than expression for the rate at which the volume is changing. 1

(iii) Find the maximum volume in the tank. 1

(iv) Find the rate when $t = 4$ and comment on your answer. 2

END OF SECTION C

SECTION D

Start a new answer booklet.

Question 7 [8 marks] **Marks**

(a) The size of a colony of ants in given by the equation $P = 5000e^{kt}$ where P is the population in t days.

(i) Show that the number of ants in the colony increases at a rate proportional to the number present. 1

(ii) If there were 6500 ants after 1 day, find the value of k to 2 decimal place. 1

(iii) On what day will the colony triple in size. 1

(b)

(i) Using the fact that $\sin 2x = 2\sin x \cos x$. Solve the equation

$$\sin 2x = 2\cos x \text{ for } 0 \le x \le 2\pi .$$ 2

(ii) Find the area between the curves $y = \sin 2x$ and $y = 2\cos x$ for $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$. 3

Question 8 [13 marks] **Marks**

(a) For the curve $y = x^3(2 - x)$.

(i) Show that it has stationary points at $(0,0)$ and $\left(\frac{3}{2}, \frac{27}{16}\right)$. 3

(ii) Determine the natures of the stationary points. 3

(iii) Hence sketch the curve showing all essential features. 2

(b) The line $y = 2 - x$ cuts the curve $y = x^3(2 - x)$ at two points A and B. The tangents to the curve at A and B intersect at C.

(i) Show that A and B are the points $(1,1)$ and $(2,0)$. 2

(ii) Find the angles that the tangents at A and B make with the positive direction of the x-axis to the nearest degree. 2

(iii) Hence find the $\angle ACB$ to the nearest degree. 1

END OF SECTION D

SECTION E

Start a new answer booklet.

Question 9 [10 marks] **Marks**

(a) Two particles A and B move along a straight line so that their displacements, in metres, from the origin at time t seconds are given by

$$x_A = 12t + 5$$
$$x_B = 6t^2 - t^3 .$$

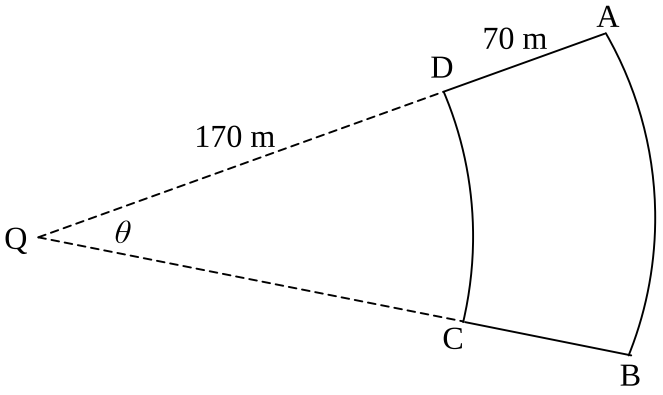
(i) Which is moving faster when $t = 1$. 2

(ii) When do the particles travel at the same speed? 2

(iii) What is the acceleration of the particle B when $t = 3$? 1

(iv) What is the maximum positive displacement of particle B? 2

(b)



In the figure, AB and CD are circular arcs which subtend an angle of θ radians at the centre Q, where $0 \le \theta \le \pi$ and AQ, DQ are radii. AD = 70m and DQ = 170m.

(i) Find expressions for the length of the arcs AB and DC. 1

(ii) A man lives at A and there is a bus stop at B, with AB, BC, CD and DA forming a road system. For what values of θ is it shorter for the man to walk along the route ADCB than along the arc AB? 2

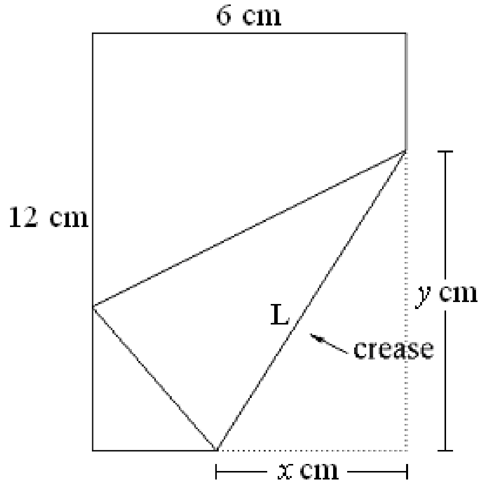
Question 10 [12 marks] **Marks**

(a) Let $f(x)=\sqrt{3}x-2\sin x$ for $0 \le x \le \pi$.

(i) Find the local maximum and/or minimum values of $f(x)$. 2

(ii) Find the greatest and least values of $f(x)$. 3

(b) A rectangular piece of paper is 12 cm high and 6 cm wide. The lower right-hand corner is folded over so as to reach the leftmost edge of the paper. L is the length of the crease and *x* and *y* are the dimensions, in centimetres, indicated on the diagram.



(i) Write an expression for the length of L in terms of *x* and *y*. 1

(ii) By considering areas or otherwise show that

$$y=\frac{x\sqrt{3}}{\sqrt{x-3}}.$$ 3

(iii) Hence are otherwise find minimal length of the crease L. 3

END OF SECTION E

END OF EXAM

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1 (a) 4.58 2DP ①

(b) $ax + 3ay - x - 3y$

$a(x+3y) - 1(x+3y)$

$(x+3y)(a-1)$ ①

(c) $\frac{d}{dx}(\ln 5x) = \frac{1}{5x} \times 5 = \frac{1}{x}$ ①

(d) $\frac{2}{(4+\sqrt{3})} \times \frac{(4-\sqrt{3})}{(4-\sqrt{3})} = \frac{2(4-\sqrt{3})}{16-3} = \frac{8-2\sqrt{3}}{13}$ ①

(e) $a + 9d = 20$
$2a + 9d = 12$

$a + 9d = 20$
$2a + 9d = 12$ −

$-a = 8$

$a = -8$

So $-8 + 9d = 20$ ①

$9d = 28$

$d = \frac{28}{9} = 3\frac{1}{9}$

(f) $\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} = 1$ ①

(g) $\int \frac{1\,dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2})$

$\int_0^1 \frac{1\,dx}{\sqrt{x^2+2^2}} = \left[\ln(x+\sqrt{x^2+4})\right]_0^1$

$= \ln(1+\sqrt{5}) - \ln(0+2) = \ln\left(\frac{1+\sqrt{5}}{2}\right)$ ②

(h) $\cos x = -\frac{1}{2}$.

quad. 2, 3.

$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$ (1)

$\pi + \frac{\pi}{3} = \frac{4\pi}{3}$ (1)

(i) $\log_x 6 = \log_x (2 \times 3)$

$= \log_x 2 + \log_x 3$

$\Rightarrow 0.6 + 0.8$

$= 1.4$ (2)

If $x^{0.6} = 2$ then $\log_x 2 = 0.6$

If $x^{0.8} = 3$ then $\log_x 3 = 0.8$

(j) $3x - 1 = 5$, $3x - 1 = -5$

$3x = 6$, $3x = -4$

$x = 2$ (1), $x = -\frac{4}{3}$ (1)

2 (a) (i) $\frac{d}{dx}(\tan x) = \sec^2 x$ ① $\quad$ /12

(ii) $\frac{d}{dx}(x^4+1)^{-2} = -2(x^4+1)^{-3} \times 4x^3$

$= \frac{-8x^3}{(x^4+1)^3}$ ①

(b) $x^2+y^2-8x+6y+9=0$

$\left(-\frac{8}{2}\right)^2$ $\quad$ $x^2-8x+16+y^2+6y+9 = -9+16+9$

$\left(+\frac{6}{2}\right)^2$ $\quad$ $(x-4)^2+(y+3)^2 = 16$

centre $(4, -3)$ ① $\quad r = 4$ ①

(c) (i) $y = \frac{k}{(x+2)} = k(x+2)^{-1}$

$\frac{dy}{dx} = -k(x+2)^{-2} \times 1 = \frac{-k}{(x+2)^2}$ ①

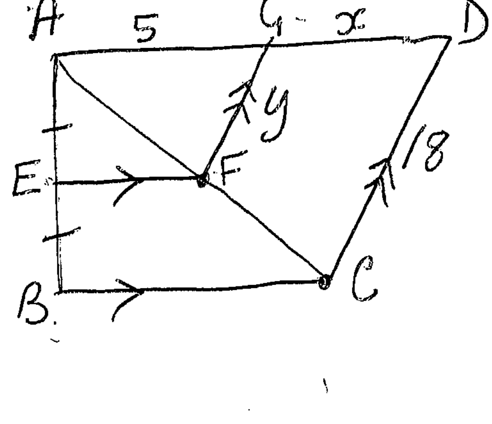
(ii) At $x=2$, $\frac{dy}{dx} = \frac{1}{4}$

$\frac{-k}{16} = \frac{1}{4}$

$-4k = 16$ ①

$k = -4$

(d)



① $x = 5$ lines cut off by equal intercepts are equal etc ① "needs" "reasons"

$y = 9$ line through the centre of 1 part of a $\triangle$ and // to a 3rd side is half its length etc

2 (e) (i) $-\frac{1}{2}\int -2\sin(2t-1)\,dt$

$= -\frac{1}{2}\cos(2t-1)+c$ (1)

(ii) $\frac{1}{2}\int_0^1 2e^{2x}\,dx$

$= \frac{1}{2}e^{2x}\Big]_0^1 = \frac{1}{2}(e^2-1)$ (1)

(f) $\log_{27}16 = x\log_3 2$

$\frac{\log_{10}16}{\log_{10}27} = \frac{\log_{10}2^x}{\log_{10}3}$

$\log_{10}2^x = \frac{\log_{10}16 \times \log_{10}3}{\log_{10}27}$

$= \frac{4\log_{10}2 \times \cancel{\log_{10}3}}{3\cancel{\log_{10}3}}$

$x\log_{10}2 = \frac{4}{3}\log_{10}2$

$x = \frac{4}{3}$ (2)

Section B

Question 3

(a) i) P has coordinates (5, 5)

C has coordinates (5, 10)

$\therefore PC = 5$ units

ii) $d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$PE = \dfrac{|4(5) + 3(5) - 10|}{\sqrt{4^2+3^2}}$

$PE = \dfrac{|25|}{5}$

$PE = 5$ units

iii) In Δ's BCP & BEP

PB is common

PC = PE (from (i) and (ii))

$\angle BCP = \angle BEP = 90°$ (given $PE \perp BE$ and $BC \perp PC$)

$\therefore \Delta BCP \equiv \Delta BEP$ (RHS)

iv) B lies on both $y = 10$ and $4x + 3y - 10 = 0$

clearly $y = 10$

when $y = 10$

$4x + 3(10) - 10 = 0$

$4x = -20$

$x = -5$

$\therefore$ B has coordinates (−5, 10)

v) BCPE is a kite

BCPE is also a cyclic quadrilateral

vi) since $\Delta BCP \equiv \Delta BEP$

area $BCPE = 2 \times \text{area } \Delta BCP$

$= \cancel{2} \times \frac{1}{\cancel{2}} \times BC \times CP$

$= 10 \times 5$

$= 50 \text{ units}^2$

vii) consider a point $Q(x, y)$ that is equidistant from BC and BE.

$$|y - 10| = \frac{|4x + 3y - 10|}{\sqrt{4^2 + 3^2}}$$

$$|y - 10| = \frac{|4x + 3y - 10|}{5}$$

$$5|y - 10| = |4x + 3y - 10|$$

$5y - 50 = 4x + 3y - 10$ or $-5y + 50 = 4x + 3y - 10$

$4x - 2y + 40 = 0$ $\quad$ $4x + 8y - 60 = 0$

$2x - y + 20 = 0$ $\quad$ $x + 2y - 15 = 0$

$\therefore x + 2y - 15 = 0$ is a locus which is equidistant from BC and BE.

(b) $\sum_{n=3}^{8} (2 \times 3^n - 2n) = 2 \times 3^3 - 2(3) + 2 \times 3^4 - 2(4) + 2 \times 3^5 - 2(5) + 2 \times 3^6 - 2(6)$
$+ 2 \times 3^7 - 2(7) + 2 \times 3^8 - 2(8)$

$= 48 + 154 + 476 + 1446 + 4360 + 13106$

$= 19590$

Question 4

(a) i) $\sin 60 = \frac{AP}{8}$

$AP = 8\sin 60$

$AP = 8\left(\frac{\sqrt{3}}{2}\right)$

$AP = 4\sqrt{3}$ cm

ii) $BQ = AP = 4\sqrt{3}$

$\tan 30 = \frac{4\sqrt{3}}{QC}$

$QC = \frac{4\sqrt{3}}{\tan 30}$

$QC = \frac{4\sqrt{3}}{\left(\frac{1}{\sqrt{3}}\right)}$

$QC = 4\sqrt{3} \times \sqrt{3}$

$QC = 12$ cm

(b) i) $P = \frac{4}{52}$

$= \frac{1}{13}$

ii) $P = \frac{26}{52}$

$= \frac{1}{2}$

iii) $P = \frac{28}{52}$ OR $P = \frac{4}{52} + \frac{26}{52} - \frac{2}{52}$ (red kings are counted twice)

$= \frac{7}{13}$

c) let $y = 0$

$3x^2 - 4x + 5 = 0$

$\Delta = b^2 - 4ac$

$\Delta = (-4)^2 - 4(3)(5)$

$\Delta = -44$

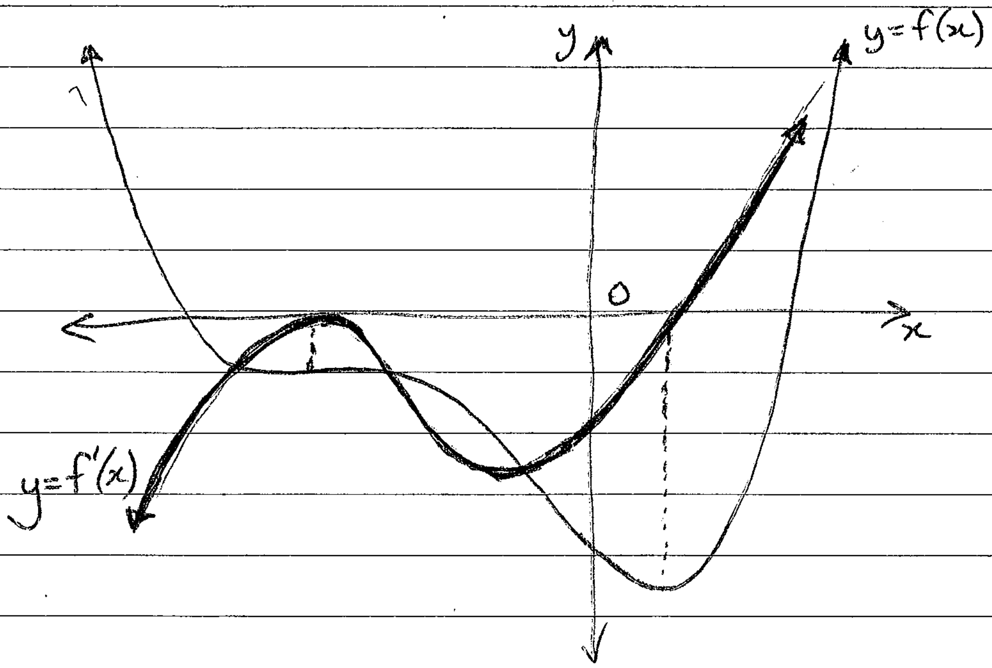
since $\Delta < 0$ the curve $y = 3x^2 - 4x + 5$ does not cut the x-axis.

(d) $\lim_{x\to 3} \frac{x^2-5x+6}{x^2-9} = \lim_{x\to 3} \frac{(x-3)(x-2)}{(x-3)(x+3)}$

$= \frac{3-2}{3+3}$

$= \frac{1}{6}$

(e)



5. (a) For the series $486 + 324 + 216 + 144 + \dots$

(i) Which term is $\dfrac{2048}{243}$? (Working must be shown.) [2]

Solution:
$$r = \tfrac{324}{486} = 2/3, \quad a = 486.$$
$$\begin{aligned} u_n = 486 \times (2/3)^{n-1} &= \frac{2048}{243}, \\ (2/3)^{n-1} &= \frac{1024}{59049}, \\ &= (2/3)^{10}. \\ n - 1 &= 10, \\ n &= 11. \end{aligned}$$
$\therefore$ It is the 11^{th} term.

(ii) Does this series have a limiting sum? Give a reason for your answer. [1]

Solution: Yes, $|r| < 1$.

(b) A polygon has 25 sides, the lengths of which form an arithmetic sequence.

(i) Find an expression for the perimeter in terms of the shortest side and the common difference. [1]

Solution:
$$\begin{aligned} \text{Put } a &= \text{shortest side}, \\ d &= \text{common difference}, \\ \text{then perimeter, } P &= \tfrac{25}{2}(2a + 24d), \\ &= 25a + 300d. \end{aligned}$$

(ii) The perimeter of the polygon is 1100 cm and the longest side is 10 times the length of the shortest side. Find the length of the shortest side of the polygon and the common difference of the sequence. [3]

Solution:
$$\begin{aligned} 10a &= a + 24d, \\ 9a &= 24d, \\ a &= \frac{8d}{3} \quad \text{——} \boxed{1} \\ 1100 &= 25a + 300d \quad \text{——} \boxed{2} \\ \text{Sub } \boxed{1} \text{ in } \boxed{2}: \quad 1100 &= \frac{200d}{3} + 300d, \\ 3300 &= 1100d, \\ d &= 3, \\ a &= 8. \end{aligned}$$
$\therefore$ The shortest side is 8 cm and the common difference is 3 cm.

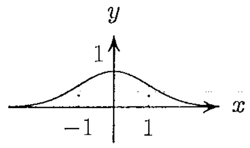
(c) For the function $y = e^{-\frac{x^2}{2}}$,

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [2]

Solution:
$$\begin{aligned}\frac{dy}{dx} &= -xe^{-x^2/2}.\\ \frac{d^2y}{dx^2} &= -e^{-x^2/2} - x\left(-xe^{-x^2/2}\right),\\ &= e^{-x^2/2}\left(x^2-1\right).\end{aligned}$$

(ii) For what values of x is the curve of $y = e^{-\frac{x^2}{2}}$ concave down? [1]

Solution:



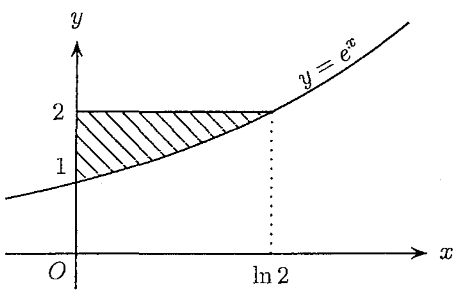
$-1 < x < 1$

(d) (i) Find the point of intersection of $y = 2$ and $y = e^x$. [1]

Solution: Equating ys, $e^x = 2 \quad \Rightarrow \quad x = \ln 2$.
$\therefore$ Intersection is at $(\ln 2,\, 2)$.

(ii) Indicate, by shading on a diagram, the region in the first quadrant bounded by the y-axis, the line $y = 2$, and the curve $y = e^x$. [1]

Solution:

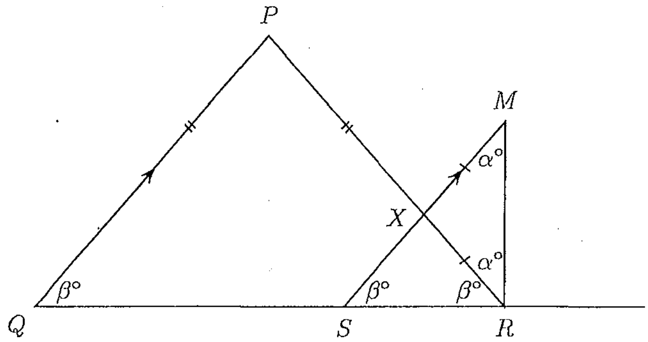


(iii) Use Simpson's Rule with 3 function values, to calculate the volume of the solid generated when the region in part (ii) is rotated about the y-axis (answer to 2 d.p.) [2]

Solution:
$$\begin{aligned}\text{Volume} &= \pi\int_1^2 x^2\,dy,\\ &= \pi\int_1^2 (\ln y)^2\,dy,\\ &\approx \pi \times \tfrac{1}{6}(0\times 1 + 0.1644\times 4 + 0.4805\times 1),\\ &\approx 0.60 \text{ (2 dec. pl.)}\end{aligned}$$

y	1	1.5	2
$(\ln y)^2$	0	0.1644	0.4805

6. (a) [3]

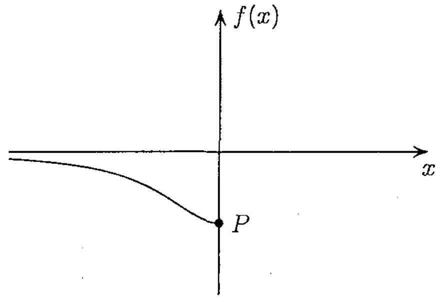


In the diagram $PQ = PR$, $XM = XR$ and $PQ \parallel MS$. By letting $\angle MRX = \alpha^\circ$, show that $MR \perp QR$, giving full reasons.

Solution:

$$\begin{aligned} \Delta XMR &\text{ is isosceles} && (XM = XR) \\ X\widehat{M}R &= M\widehat{R}X = \alpha^\circ && (\text{base } \angle\text{s of isosceles } \Delta XMR) \\ \Delta PQR &\text{ is isosceles} && (PQ = PR) \\ P\widehat{Q}R &= Q\widehat{R}P = \beta^\circ && (\text{base } \angle\text{s of isosceles } \Delta PQR) \\ M\widehat{S}R &= P\widehat{Q}R = \beta^\circ && (\text{corresponding } \angle\text{s, } PQ \parallel MS) \\ 180^\circ &= 2\alpha^\circ + 2\beta^\circ && (\angle \text{ sum of } \Delta MSR) \\ \therefore\ \alpha^\circ + \beta^\circ &= 90^\circ, \\ i.e.\ M\widehat{R}S &= 90^\circ. \\ \therefore\ MR &\perp QR. \end{aligned}$$

(b) The diagram below shows part of the function $f(x) = \dfrac{-2}{1+x^2}$.



(i) Find the coördinates of point P on the diagram. [1]

Solution: $f(0) = -2$.
$\therefore\ P(0, -2)$.

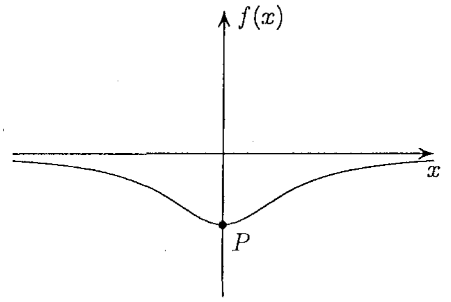
(ii) Prove that $f(x)$ is an even function. [1]

Solution:
$$\begin{aligned} f(-x) &= \frac{-2}{1+(-x)^2}, \\ &= \frac{-2}{1+x^2}, \\ &= f(x). \end{aligned}$$

$\therefore\ f(x)$ is even.

(iii) Copy the diagram into your answer booklet and complete the curve. [1]

Solution:



(iv) State the range of this function. [1]

Solution: $-2 \leq f(x) < 0$.

(c) The volume V litres, of a tank after t minutes is given by $V = 60 + 4t - t^2$.

(i) At what time(s) will the tank be empty? [1]

Solution:
$$\begin{aligned} V &= 60 + 4t - t^2, \\ &= 60 + 10t - 6t - t^2, \\ &= 10(6+t) - t(6+t), \\ &= (10-t)(6+t), \\ &= 0 \text{ when } t = -6, 10. \end{aligned}$$

P -60
S 4
F $10, -6$

$\therefore$ The tank *will* be empty after 10 minutes (and was empty 6 minutes before observation started, although this is not required by the question).

(ii) Find an expression for the rate at which the volume is changing. [1]

Solution: $\dfrac{dV}{dt} = 4 - 2t$.

(iii) Find the maximum volume in the tank. [1]

Solution:
$$\frac{dV}{dt} = 0 \text{ when } t = 2.$$
$$\frac{d^2V}{dt^2} = -2.$$
$$\begin{aligned} \therefore \text{Max. volume (when } t = 2) &= 60 + 8 - 4, \\ &= 64 \text{ litres.} \end{aligned}$$

(iv) Find the rate when $t = 4$ and comment on your answer. [2]

Solution: When $t = 4$, $\frac{dV}{dt} = 4 - 8 = -4$.
This means the tank is emptying at 4 L/min.

SECTION D

Question 7

(a) $P = 5000e^{kt}$

(i) $\dfrac{dP}{dt} = 5000ke^{kt}$

$= k\left[5000e^{kt}\right]$ 1

$\dfrac{dP}{dt} = kP$

(ii) $\left.\begin{array}{l} t=1 \\ P=6500 \end{array}\right\}$ $P = 5000e^{kt}$

$6500 = 5000e^{k}$

$k = \ln\left(\frac{13}{10}\right)$ 1

$k \doteqdot 0.26$

(iii) $P = 5000e^{0.26t}$

$15000 = 5000e^{0.26t}$

$3 = e^{0.26t}$

$t = \frac{1}{0.26}\ln 3$ 1

$\Rightarrow t \doteqdot 4.23$

During the 5th day

(b)

(i)

$\sin 2x = 2\cos x$

$2\sin x\cos x = 2\cos x$

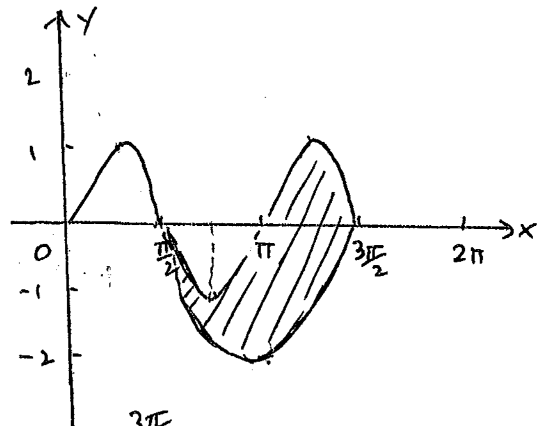
$2\cos x\left[\sin x - 1\right] = 0$ 2

$\cos x = 0$ or $\sin x - 1 = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$, $x = \frac{\pi}{2}$

$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$

(ii)



$\text{Area} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[\sin 2x - 2\cos x\right] dx$ 3

$= \left[-\frac{1}{2}\cos 2x - 2\sin x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$

$= \left[\left(-\frac{1}{2}\cos 3\pi - 2\sin\frac{3\pi}{2}\right) - \left(-\frac{1}{2}\cos\pi - 2\sin\frac{\pi}{2}\right)\right]$

$= \left[-\frac{1}{2}(-1) - 2(-1)\right] + \frac{1}{2}(-1) + 2$

$= 4 \text{ units}^2$

Question 8

(a) $y = x^3(2-x)$

(i) $y = 2x^3 - x^4$ 3

$\frac{dy}{dx} = 6x^2 - 4x^3 = 0$

when $x = 0$

or $x = 3/2$

$[0,0]$ $[3/2, 27/16]$

(ii) $\frac{d^2y}{dx^2} = 12x - 12x^2$

When $x = 0$, $y'' = 0$ 3

$\Rightarrow$ possible P.O.I.

x	<	0	>
y''	−	0	+

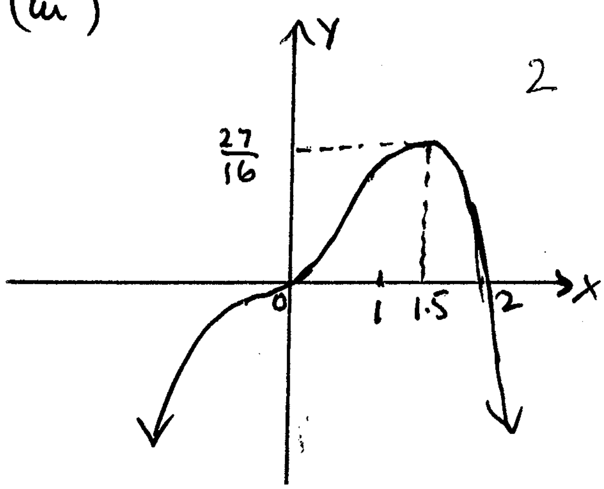
Change of concavity

$\Rightarrow$ Horiz. P.O.I at $(0,0)$

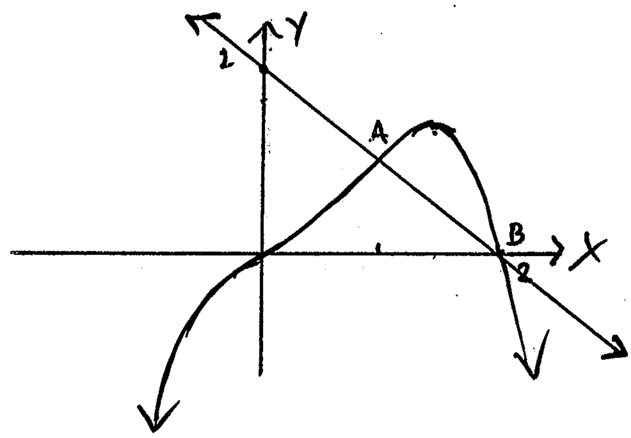
When $x = \frac{3}{2}$, $y'' = -9 < 0$

$\Rightarrow$ Max T.P. at $(\frac{3}{2}, \frac{27}{16})$

(iii) 2



(b)



(i) Solve simultaneously (or simply substitute pts)

$\Rightarrow 2 - x = x^3(2-x)$ 2

$(2-x)\left[1 - x^3\right] = 0$

$\therefore \begin{pmatrix} x = 2 \\ y = 0 \end{pmatrix}$ or $\begin{pmatrix} x = 1 \\ y = 1 \end{pmatrix}$

(ii) <u>grad. tangent at A (1,1)</u>

$\frac{dy}{dx} = 6x^2 - 4x^3$ 2

At $x = 1$, grad. $= 2$

$\tan\alpha = 2 \Rightarrow \alpha = 63°$

<u>grad. tangent at B (2,0)</u>

$\frac{dy}{dx} = 6x^2 - 4x^3$

At $x = 2$, grad $= -8$

$\tan\theta = -8 \Rightarrow \theta = 97°$

(iii) $63° + A\hat{C}B = 97°$ 1

$A\hat{C}B = 34°$

Q9. (a) $x_A = 12t + 5$ $\quad$ $x_B = 6t^2 - t^3$

$\dot{x}_A = 12$ $\quad$ $\dot{x}_B = 12t - 3t^2$

(i) when $t = 1$ $\quad$ $\dot{x}_A = 12$ & $\dot{x}_B = 12 - 3 = 9$

$\therefore$ x_A ✓✓

(ii) let $12 = 12t - 3t^2$

$3t^2 - 12t + 12 = 0$

$3(t^2 - 4t + 4) = 0$

$3(t-2)^2 = 0$

$t = 2$. $\quad$ $\therefore$ after 2 seconds ✓✓

(iii) $\ddot{x}_B = 12 - 6t$.

$\therefore$ when $t = 3$

$\ddot{x}_B = 12 - 18$

$= -6$. $\quad$ $\therefore$ $-6\ m\,s^{-2}$ ✓

(iv) let $\dot{x}_B = 0$ ie $12t - 3t^2 = 0$

$3t(4 - t) = 0$

$\therefore t = 0, 4$.

$\therefore$ after 4 secs. $x_B = 6 \times 16 - 64$

$= 96 - 64$

$= 32\ m$

✓✓

9 (CONT'D)

(b) (i) $AB = 240\theta$.

$DC = 170\theta$.

✓

(ii). let $70 + 170\theta + 70 < 240\theta$.

$140 < 70\theta$.

$\theta > \frac{140}{70}$

$\theta > 2$.

∴ $2 < \theta \leq \pi$.

✓✓

Q10. (a)(i) $f(x) = \sqrt{3}x - 2\sin x \quad , 0 \le x \le \pi$.

$f'(x) = \sqrt{3} - 2\cos x$.

$f''(x) = 2\sin x$.

need for max/min $f'(x) = \sqrt{3} - 2\cos x = 0$

$2\cos x = \sqrt{3}$

$\cos x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{6}$.

$\therefore y = \frac{\sqrt{3}.\pi}{6} - 2\sin\frac{\pi}{6}$.

$= \frac{\sqrt{3}\pi}{6} - 2 \times \frac{1}{2}$.

$= \frac{\sqrt{3}\pi}{6} - 1$.

Test $f''(\frac{\pi}{6}) = 2 \times \sin\frac{\pi}{6}$

$= 2 \times \frac{1}{2}$

$= 1$.

✓✓

$\therefore \frac{\sqrt{3}\pi}{6} - 1$ is a local MIN.

(ii). Consider $f(0) = \sqrt{3}.0 - 2\sin 0$

$= 0$. ✓

& $f(\pi) = \sqrt{3}\pi - 2\sin\pi$.

$= \sqrt{3}\pi - 0$

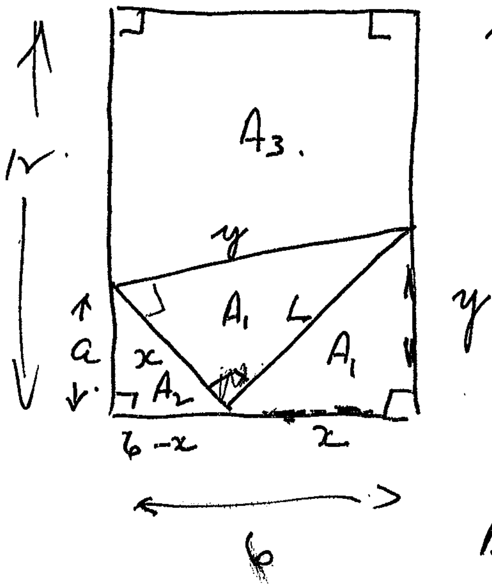
$= \sqrt{3}\pi$ ✓

$\therefore$ GREATEST VALUE is $\sqrt{3}\pi$. ✓

& LEAST VALUE is $\frac{\sqrt{3}\pi}{6} - 1$. (NB ≈ 0.093.)

Q10. (b) (i) $L = \sqrt{x^2 + y^2}$ ✓

(ii)



Using Pythagoras

$$a^2 = x^2 - (6-x)^2$$
$$= x^2 - (36 - 12x + x^2)$$
$$a^2 = 12x - 36.$$
$$a = \sqrt{12x - 36}.$$
$$= 2\sqrt{3x - 9}.$$

By considering areas.

$$2A_1 + A_2 + A_3 = 72.$$

where $A_1 = \frac{xy}{2}$

$$A_2 = \frac{2\sqrt{3x-9}\,(6-x)}{2}$$
$$= (6-x)\sqrt{3x-9}$$

& $A_3 = \frac{1}{2} \times 6 \times \left[(12-y) + 12 - 2\sqrt{3x-9}\right]$

$$\therefore xy + (6-x)\sqrt{3x-9} + 3\left[(12-y) + 12 - 2\sqrt{3x-9}\right] = 72$$

$$xy + 6\sqrt{3x-9} - x\sqrt{3x-9} + 3\left[24 - y - 2\sqrt{3x-9}\right] = 72$$

$$\therefore xy + 6\sqrt{3x-9} - x\sqrt{3x-9} + 72 - 3y - 6\sqrt{3x-9} = 72.$$ ✓✓✓

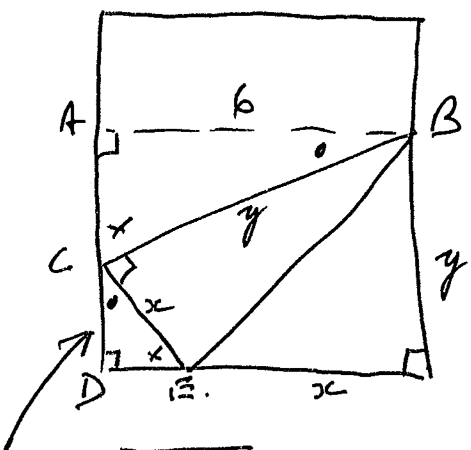
$$\therefore xy - x\sqrt{3x-9} + 72 - 3y = 72.$$

$$y(x-3) = x\sqrt{3x-9}$$

$$y = \frac{x\sqrt{3}\sqrt{(x-3)}}{x-3}$$

$$\therefore y = \frac{\sqrt{3}\,.\,x}{\sqrt{x-3}}.$$

OR Using Pythagoras



$a = 2\sqrt{3x-9}$

$$y^2 = 36 + \left(y - 2\sqrt{3x-9}\right)^2$$

$$y^2 = 36 + y^2 - 4y\sqrt{3x-9} + 4(3x-9)$$

$$y^2 = 36 + y^2 - 4y\sqrt{3x-9} + 12x - 36.$$

$$4y\sqrt{3x-9} = 12x.$$

$$y\sqrt{3x-9} = 3x.$$

$$y = \frac{3x}{\sqrt{3x-9}}$$

$$= \frac{3x}{\sqrt{3}\sqrt{x-3}}$$

$$\therefore y = \frac{\sqrt{3}x}{\sqrt{x-3}}$$

OR Establish that $\triangle ABC \,|||\, \triangle DCE$.

then.
$$\frac{y}{6} = \frac{x}{2\sqrt{3x-9}}$$

$$y = \frac{3x}{\sqrt{3}\sqrt{x-3}}$$

$$y = \frac{x\sqrt{3}}{\sqrt{x-3}}.$$

Q10 (CONTD)

(iii)

$$L^2 = x^2 + y^2$$

$$= x^2 + \left(\frac{\sqrt{3x}}{\sqrt{x-3}}\right)^2$$

$$= x^2 + \frac{3x^2}{x-3}$$

$$= \frac{x^2(x-3)+3x^2}{x-3}$$

$$= \frac{x^3-3x^2+3x^2}{x-3}$$

$$= \frac{x^3}{x-3}.$$

now L will be a min. when L^2 is a min.

$$\frac{d}{dx}(L^2) = \frac{(x-3)3x^2 - x^3}{(x-3)^2}$$

$$= \frac{3x^3-9x^2-x^3}{(x-3)^2}$$

$$= \frac{2x^3-9x^2}{(x-3)^2}$$

Let $\frac{d}{dx}(L^2) = 0$ $\therefore$ $2x^3-9x^2=0$

$$x^2(2x-9)=0$$

$$x = 0, \frac{9}{2}$$

clearly $x \neq 0$ $\therefore$ at $x = \frac{9}{2}$ $L^2 = \frac{\left(\frac{9}{2}\right)^3}{9/2 - 3}$

$$= \frac{9^3}{3/2 \times 8}$$

$$= \frac{3 \times 9^2}{4}$$

$$\therefore L = \frac{9\sqrt{3}}{2} \quad (\sim 7.79)$$

Test

x	4	$4\frac{1}{2}$	5
$\frac{d}{dx} L^2$	-16	0	$25/4$
	\	—	/

$\therefore$ $L = \frac{9\sqrt{3}}{2}$ is a <u>MINIMUM</u>.

✓✓✓